JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 20, No. 6, November–December 1997

Differential Game Approach to the Mixed H_2–H_∞ Problem

Gregory D. Sweriduk* and Anthony J. Calise†
Georgia Institute of Technology, Atlanta, Georgia 30332

H_∞ control theory has gained wide acceptance over the past 10 years as a valuable method of controller design. However, for most applications the solution results in a controller of higher dimension than that of the plant. As an alternative to controller order reduction, low-order controllers can be designed by fixing the order of the controller a priori. There also has been a growing interest in the mixed H_2–H_∞ problem. This problem has not been resolved completely, and there are few published solutions for the fixed-order case. A differential game formulation for the H_∞ problem is extended to address the mixed problem. The result is five first-order necessary conditions, and a conjugate gradient algorithm is used to search for solutions. To test the approach, a 10th-order problem describing the lateral dynamics of a high-performance helicopter in forward flight is used.

Introduction

H_∞ control theory has provided a powerful technique for designing robust controllers. One disadvantage is that it produces high-order controllers. To obtain the desired frequency characteristics for the closed-loop system, it is necessary to include frequency-dependent weights in the design problem. Once the weights have been included, the problem will be higher in order than the nominal plant to be controlled, and hence the controller will be of correspondingly higher order. Designing for robust performance or structured uncertainty further increases the controller dimension. When implemented, large-order controllers can create time delays that may be undesirable. One solution to this problem is to use model order reduction on the controller realization.[1] However, existing methods of order reduction that do not consider the properties of the closed-loop system are not guaranteed to preserve the performance and robustness properties of the full-order design. Another approach to designing low-order compensators is to constrain the order of the compensator in the design process. The concept of fixed-order dynamic compensators was introduced in the early 1970s for the H_2 control problem. Applications to robust control have been pioneered mainly by Bernstein and Haddad[2] using the concept of optimal projection. Controller synthesis requires the solution of two modified Riccati equations and two Lyapunov equations, all coupled through a projection operator.

The objective of the mixed H_2–H_∞ problem is to minimize the H_2 norm of one transfer function while satisfying an upper bound on the H_∞ norm of another transfer function. Realistically, the value lies in being able to study the tradeoff between performance and robustness for a given design problem. As pointed out previously,[3] the H_2 norm makes more sense for performance, whereas the H_∞ norm makes more sense for robustness. The true mixed problem has two inputs and two outputs because the classes of disturbances, as well as the performance variables, are different. Research has been conducted using variations of this problem in which there is only one input or only one output. Bernstein and Haddad[2] consider the case of two outputs and one input, with both full-order and fixed-order control. The necessary conditions are a variant of the optimal projection equations of Hyland and Bernstein.[4] A dual problem is addressed in which there are two inputs (one white noise and the other bounded energy) and one output.[3,5] Necessary and sufficient conditions for the existence of a controller, which include a set of equations similar to those of Bernstein and Haddad,[2] have been derived. The two-output, one-input problem that assumes full-order control was considered, and a convex optimization problem has been developed.[6] The difficulty with these synthesis methods is the gap that may exist between the H_∞ overbound that is used in the formulation and the true infinity norm of the synthesized controller. Rotea and Khargonekar[7] addressed the two-input, two-output case with full-state feedback assumed. The assumption of perfect-state information was removed in other research,[8,9] which also treats the fixed-order problem but does not attempt to solve it. In another case,[10] the order is fixed to be that of the H_2 controller. In the present paper, a straightforward modification of a differential game formulation of the H_∞ problem is used to obtain fixed-order controllers for the true H_2–H_∞ problem.

Linear quadratic differential games[11,12] were explored long before the development of H_∞ theory. As pointed out in the literature, these results solve the full-state H_∞ problem almost completely.[13] Rhee and Speyer[14] develop the differential game approach for the incomplete-information H_∞ problem with no constraint on the compensator order, which results in the two-Riccati-equation solution developed elsewhere[15]; similar results were obtained by Basar and Bernhard.[16] El Ghaoui et al.[17] also considered the game theory approach for the full-order H_∞ problem and suggested but did not solve the fixed-order compensator problem. A modified version of the game theory approach was developed for the problem of fixed-order dynamic compensation.[18] This resulted in a set of first-order necessary conditions for the existence of an upper value for the game. This paper extends those results[18] to include the true mixed H_2–H_∞ problem. In Ref. 19, the controller minimizes the worst-case ratio between the resultant H_2 performance degradation and the performance degradation that occurs with the standard H_∞ controller. A key feature of our method is the use of a particular canonical form for the compensator. This allows the necessary conditions to be expressed in the form of five matrix equations. A conjugate gradient algorithm is outlined that can be used to find solutions. An example is presented to illustrate the synthesis procedure.

Problem Statement

In this section a set of first-order necessary conditions is derived. First, the H_∞ problem is reviewed in the context of game theory, and then some properties regarding the H_2 norm are presented. The mixed problem then is formulated using the cost function from the H_∞ problem and a constraint from the H_2 problem.

H_∞ Problem

Consider a realization of the generalized plant given by

$$\dot{x} = Ax + B_1 w + B_2 u \tag{1}$$

$$z = C_1 x + D_{12} u \tag{2}$$

$$y = C_2 x + D_{21} w + D_{22} u \tag{3}$$

Received June 13, 1994; presented as Paper 94-3660 at the AIAA Guidance, Navigation, and Control Conference, Scottsdale, AZ, Aug. 1–3, 1994; revision received June 4, 1997; accepted for publication June 18, 1997.

*Research Assistant, School of Aerospace Engineering; currently Research Scientist, Optimal Synthesis, Palo Alto, CA 94306.
†Professor, School of Aerospace Engineering. Fellow AIAA.

where $x \in R^n$ is the state vector, $w \in R^{m1}$ is the disturbance vector, $u \in R^{m2}$ is the control vector, $z \in R^{p1}$ is the performance vector, and $y \in R^{p2}$ is the observation vector. It is also assumed that

$$(A, B_1, C_1) \text{ is stabilizable and detectable}$$

$$(A, B_2, C_2) \text{ is stabilizable and detectable}$$

and that D_{12} has full column rank and D_{21} has full row rank. D_{11} is assumed to be zero so that the problem is well posed.

The compensator is defined in controller canonical form[20]:

$$\dot{x}_c = P^0 x_c + N^0 u_c - N^0 y \tag{4}$$

$$u_c = -P x_c \tag{5}$$

$$u = -H x_c \tag{6}$$

where $x_c \in R^{nc}$ and $u_c \in R^{p2}$. P and H are free-parameter matrices, and P^0 and N^0 are fixed matrices of zeros and ones determined by the choice of controllability indices. When $nc = n$, the compensator is full order but the realization is different from the observer-based solution in Ref. 15. However, if a similarity transformation exists, the canonical compensator and the standard solution are equivalent.[21]

To construct the closed-loop system, let

$$\bar{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}, \qquad \bar{u} = \begin{bmatrix} u \\ u_c \end{bmatrix} \tag{7}$$

The augmented system is

$$\begin{aligned}\dot{\bar{x}} &= \begin{bmatrix} A & 0 \\ -N^0 C_2 & P^0 \end{bmatrix} \bar{x} + \begin{bmatrix} B_1 \\ -N^0 D_{21} \end{bmatrix} w + \begin{bmatrix} B_2 & 0 \\ -N^0 D_{22} & N^0 \end{bmatrix} \bar{u} \\ &= \bar{A}\bar{x} + \bar{B}_1 w + \bar{B}_2 \bar{u}\end{aligned} \tag{8}$$

$$\bar{y} = x_c = [0 \quad I] \begin{bmatrix} x \\ x_c \end{bmatrix} = \bar{C}_2 \bar{x} \Rightarrow \bar{u} = -G\bar{y}, \quad \text{where} \quad G = \begin{bmatrix} H \\ P \end{bmatrix} \tag{9}$$

$$\bar{C}_1 = [C \quad 0], \qquad \bar{D}_{12} = [D_{12} \quad 0] \tag{10}$$

Equations (8) and (9) define a static gain output feedback problem. The closed-loop system (Fig. 1) is

$$\dot{\bar{x}} = (\bar{A} - \bar{B}_2 G \bar{C}_2)\bar{x} + \bar{B}_1 w = \tilde{A}\bar{x} + \tilde{B} w \tag{11}$$

where

$$\tilde{A} = \begin{bmatrix} A & -B_2 H \\ -N^0 C_2 & P^0 - N^0 P - N^0 D_{22} H \end{bmatrix} \tag{12}$$

and

$$\bar{x}(0) = \begin{bmatrix} x(0) \\ x_c(0) \end{bmatrix} \tag{13}$$

The objective is to minimize the infinity norm of the transfer function from w to z, denoted by $T_{zw}(s)$. If $z(s) = T_{zw}(s)w(s)$, then

$$\|T_{zw}\|_\infty = \sup_{w \in L_2} \frac{\|T_{zw} w\|_2}{\|w\|_2} = \sup_{\|w\|_2 \le 1} \|T_{zw} w\|_2 \tag{14}$$

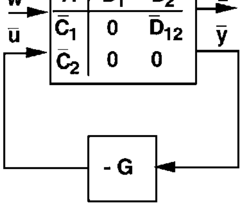

Fig. 1 Closed-loop system in constant gain output feedback form.

The optimization problem, in mathematical terms, is to find

$$\inf\{\|T_{zw}(G)\|_\infty : G \in \boldsymbol{G}\} := \gamma^* \tag{15}$$

where $\boldsymbol{G} = \{G \in R^{(m+p)xnc} : \tilde{A} \text{ is stable}\}$. It is assumed that this set is nonempty. Because of computational problems associated with trying to achieve the infimum, a more practical design objective would be to find a G that ensures $\|T_{zw}\|_\infty^2 \le \gamma^2$ for some $\gamma > \gamma^*$. Note that by defining

$$\tilde{C} = \bar{C}_1 - \bar{D}_{12} G \bar{C}_2 = [C_1 \quad -D_{12} H] \tag{16}$$

then

$$T_{zw}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} \tag{17}$$

Equations (14) and (15) suggest a min-max problem using the cost functional

$$J_\gamma(w, G) = E\left\{\int_0^\infty (z^T z - \gamma^2 w^T w)\,\mathrm{d}t\right\} \tag{18}$$

where $E\{\cdot\}$ is the expectation operator, which is taken over a distribution of initial conditions in Eq. (13), assuming zero mean and variance X_0. This functional differs from what was used in Refs. 14 and 16 in the treatment of the effect of initial conditions. Rather than optimizing the initial condition, a fixed distribution is used. Note that, if $x(0) = 0$ and $\|T_{zw}\|_\infty^2 \le \gamma^2$, it follows then from Eqs. (14) and (18) that $\|z\|_2^2 \le \gamma^2 \|w\|_2^2$, $J_\gamma = \|z\|_2^2 - \gamma^2\|w\|_2^2 \le 0$, and $J = 0$ only for $w^* = 0$. This explains the need for introducing the distribution X_0. If $X_0 = \tilde{B}\tilde{B}^T$, then $J_\gamma(0, G) = \|T_{zw}\|_2^2$. That is, X_0 may be viewed as the consequence of applying impulse functions at the disturbance inputs at $t = 0$, whose strengths are uniformly distributed on the unit sphere.

In the framework of differential games, min-max J_γ (vs max-min) is called the upper value of the game. The upper value is defined only over the set

$$\boldsymbol{G}_\gamma = \left\{G \in R^{(m+p)xnc} : \tilde{A} \text{ is stable} \quad \text{and} \quad \|T_{zw}\|_\infty^2 \le \gamma^2\right\}$$

where $\gamma > \gamma^*$. The min-max problem is to find the disturbance w^* that maximizes the cost functional J_γ and the control G^* that minimizes $J_\gamma(w^*, G)$, which together result in the upper value

$$J_\gamma^* = \min_{G \in \boldsymbol{G}_\gamma} \max_{w \in L_2} \{J_\gamma(w, G)\} \tag{19}$$

The optimization is subject to the dynamic constraints in Eq. (11). The choice of X_0 is useful if the solution of the problem defined in Eq. (19) is to reduce to the standard H_2 problem in the limit as g approaches infinity. It also appears to play a crucial role in ensuring a finite minimizing solution for G for a given value of $\gamma > \gamma^*$.

Define

$$Q_s = \begin{bmatrix} C_1^T C_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad R_s = \begin{bmatrix} D_{12}^T D_{12} & 0 \\ 0 & 0 \end{bmatrix}$$
$$S_s = \begin{bmatrix} D_{12}^T C_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{20}$$

Equation (18) now can be expressed in terms of the states as

$$J_\gamma = E\left\{\int_0^\infty (\bar{x}^T \tilde{C}^T \tilde{C} \bar{x} - \gamma^2 w^T w)\,\mathrm{d}t\right\} \tag{21}$$

where

$$\tilde{C}^T \tilde{C} = Q_s + \bar{C}_2^T G^T R_s G \bar{C}_2 - \bar{C}_2^T G^T S_s - S_s^T G \bar{C}_2 \tag{22}$$

Worst-Case Disturbance

If $G \in \boldsymbol{G}_\gamma$, it follows from Lemmas 4 and 1 of Ref. 15 that there exists a positive semidefinite solution to the algebraic Riccati equation

$$\tilde{A}^T Q_\infty + Q_\infty \tilde{A} + \tilde{C}^T \tilde{C} + \gamma^{-2} Q_\infty \tilde{B}\tilde{B}^T Q_\infty = 0 \tag{23}$$

and that $\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^T Q_\infty$ is stable. It immediately follows that, for any $G \in \boldsymbol{G}_\gamma$, a unique maximizing solution exists and is given by[15,16]

$$w = \gamma^{-2}\tilde{B}^T Q_\infty \bar{x} = K_\infty \bar{x} \tag{24}$$

where Q_∞ is the minimal positive semidefinite solution of Eq. (23). Thus, the worst-case disturbance is a feedback of the states, and from Lemma 12 of Ref. 15, it can be shown that $w \in L_2$. Using Eq. (24) in Eq. (21) allows the cost function to be expressed solely in terms of the states

$$J_\gamma = E\left\{\int_0^\infty \bar{x}^T\left[\tilde{C}^T\tilde{C} - \gamma^{-2}Q_\infty\tilde{B}\tilde{B}^T Q_\infty\right]\bar{x}\,\mathrm{d}t\right\} \tag{25}$$

The cost in Eq. (25) can be expressed as

$$J_\gamma = E\left\{\bar{x}^T(0)Q_\infty\bar{x}(0)\right\} = \mathrm{tr}[X_0 Q_\infty] \tag{26}$$

Necessary Conditions

By adjoining Eq. (23) to Eq. (26), the constrained optimization problem can be expressed with the Lagrangian

$$L = \mathrm{tr}\Big[\sigma Q_\infty X + \big(\tilde{A}^T Q_\infty + Q_\infty\tilde{A} + \tilde{C}^T\tilde{C} + \gamma^{-2}Q_\infty\tilde{B}\tilde{B}^T Q_\infty\big)\boldsymbol{L}\Big] \tag{27}$$

where $\boldsymbol{L}$ is a Lagrange multiplier matrix. The first-order necessary conditions for optimality follow from the matrix gradient conditions[18]:

$$\frac{\partial L}{\partial Q_\infty} = 0, \qquad \frac{\partial L}{\partial \boldsymbol{L}} = 0, \qquad \frac{\partial L}{\partial G} = 0, \qquad Q_\infty \geq 0, \qquad \boldsymbol{L} \geq 0 \tag{28}$$

H_∞ controllers are synthesized by solving Eq. (28) for successively lower values of γ. Each solution is internally stabilizing and satisfies the condition.

Existence of a Minimizing Solution

The usual method of proof is to show that the cost function is continuous on a closed and bounded subset of the open set $\boldsymbol{G}_\gamma$ and that the minimum occurs on the interior of the subset.[22] In H_2 problems, this is straightforward because it can be shown that, under a mild set of assumptions, the cost function is unbounded both on the boundary of the set of stabilizing gains and when any element in the feedback gain matrix becomes unbounded. Such is not the case here because the solution to Eq. (23) normally remains bounded as the boundary of the set $\boldsymbol{G}_\gamma$ is approached. However, all test cases to date have shown that J_γ increases near the boundary, yielding a minimum in the interior of $\boldsymbol{G}_\gamma$. Unfortunately, the set of conditions that ensures this property has not been determined, although the detectability assumptions play a crucial role. Also, it can be said that the choice of X_0 is also important to the existence of a finite minimum for G.

Mixed H_2–H_∞ Problem

The H_2 norm of a transfer function is defined as

$$\|G(j\omega)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{tr}[G^*(j\omega)G(j\omega)]\,\mathrm{d}\omega \tag{29}$$

where $G(s) = C(sI - A)^{-1}B$. It is well known that the H_2 norm can be expressed in terms of the controllability or observability gramians:

$$\|G(j\omega)\|_2^2 = \mathrm{tr}\big(C\boldsymbol{L}_c C^T\big) = \mathrm{tr}\big(B^T\boldsymbol{L}_0 B\big) \tag{30}$$

where

$$\tilde{A}\boldsymbol{L}_c + \boldsymbol{L}_c\tilde{A}^T + BB^T = 0 \tag{31}$$

$$\tilde{A}^T\boldsymbol{L}_0 + \boldsymbol{L}_0\tilde{A}^T + C^T C = 0 \tag{32}$$

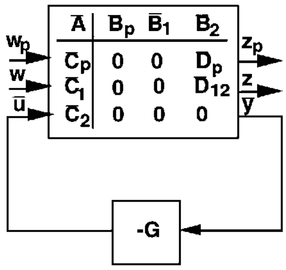

Fig. 2 Mixed H_2/H_∞ problem.

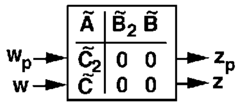

Fig. 3 Closed-loop system for mixed problem.

Consider the system shown in Eqs. (1–3) but with the addition of another disturbance (white noise) and another controlled output:

$$\dot{x} = Ax + B_1 w + B_p w_p + B_2 u \tag{33}$$

$$z = C_1 x + D_{12} u \tag{34}$$

$$z_p = C_p x + D_{1p} u \tag{35}$$

$$y = C_2 x + D_{21} w + D_{2p} w_p + D_{22} u \tag{36}$$

The corresponding static gain output feedback formulation is depicted in Fig. 2, where $\bar{A}$, $\bar{B}_1$, $\bar{B}_2$, $\bar{C}_1$, $\bar{C}_2$, and $\bar{D}_{12}$ are defined as before and $\bar{B}_p$, $\bar{C}_p$, and $\bar{D}_p$ are

$$\bar{B}_p = \begin{bmatrix} B_p \\ -N^0 D_{2p}\end{bmatrix}, \qquad \bar{C}_p = [C_p \quad 0], \qquad \bar{D}_p = [D_{1p} \quad 0]$$

In the closed-loop system shown in Fig. 3, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are defined in Eqs. (11), (12), and (16) and $\tilde{B}_2$ and $\tilde{C}_2$ are

$$\tilde{B}_2 = \bar{B}_p, \qquad \tilde{C}_2 = \bar{C}_p - \bar{D}_p G\bar{C}_2 \tag{37}$$

The H_2 norm of $T_{z_p w_p}(s) = \tilde{C}_2(sI - \tilde{A})^{-1}\tilde{B}_2$ can be expressed using either form of Eq. (30). It is therefore straightforward to include an H_2 criterion in Eq. (26). The H_2 norm and its corresponding Lyapunov equation are adjoined to the Lagrangian in Eq. (27).

Using the observability form in Eqs. (30) and (32), we have

$$J'_\gamma = \mathrm{tr}(X_0 Q_\infty) + \lambda\,\mathrm{tr}\big(X\tilde{C}_2^T\tilde{C}_2\big) \tag{38}$$

where X satisfies

$$\tilde{A}X + X\tilde{A}^T + \tilde{B}_2\tilde{B}_2^T = 0 \tag{39}$$

The Lagrangian in Eq. (27) becomes, for the mixed problem,

$$L = \mathrm{tr}\Big[\sigma Q_\infty X + \big(\tilde{A}^T Q_\infty + Q_\infty\tilde{A} + \tilde{C}^T\tilde{C} + \gamma^{-2}Q_\infty\tilde{B}\tilde{B}^T Q_\infty\big)\boldsymbol{L} + \lambda X\tilde{C}_2^T C_2 + \big(\tilde{A}X + X\tilde{A}^T + \tilde{B}_2\tilde{B}_2^T\big)\boldsymbol{L}_2\Big] \tag{40}$$

where $\boldsymbol{L}$ and $\boldsymbol{L}_2$ are Lagrange multiplier matrices. A scalar weight λ on the H_2 norm allows a tradeoff between performance (the H_2 norm) and robustness (the H_∞ norm).

Taking gradients, the first-order necessary conditions are

$$\frac{\partial L}{\partial Q_\infty} = \sigma X_0 + \big(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^T Q_\infty\big)\boldsymbol{L} + \boldsymbol{L}\big(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^T Q_\infty\big) = 0 \tag{41}$$

$$\frac{\partial L}{\partial \boldsymbol{L}} = \tilde{A}^T Q_\infty + Q_\infty\tilde{A} + \tilde{C}^T\tilde{C} + \gamma^{-2}Q_\infty\tilde{B}\tilde{B}^T Q_\infty = 0 \tag{42}$$

$$\frac{\partial L}{\partial X} = \tilde{A}^T \boldsymbol{L}_2 + \boldsymbol{L}_2 \tilde{A} + \lambda \tilde{C}_2^T \tilde{C}_2 = 0 \tag{43}$$

$$\frac{\partial L}{\partial \boldsymbol{L}_2} = \tilde{A} X + X \tilde{A}^T + \tilde{B}_2 \tilde{B}_2^T = 0 \tag{44}$$

$$\frac{\partial L}{\partial X} = 2\big(R_s G \bar{C}_2 \boldsymbol{L} \bar{C}_2^T - S_s \boldsymbol{L} \bar{C}_2^T - \bar{B}_2^T Q_\infty \boldsymbol{L} \bar{C}_2^T$$

$$+ \lambda R_2 G \bar{C}_2 X \bar{C}_2^T - \lambda S_2 X \bar{C}_2^T - \bar{B}_2^T \boldsymbol{L}_2 X \bar{C}_2^T\big) \tag{45}$$

This formulation is mathematically equivalent to finding a controller such that $G \in \boldsymbol{G}_\gamma$, subject to a constraint on the H_2 norm, because it can be shown that there exists a monotonic relationship between λ and the resulting H_2 norm constraint.[23] However, rather than specify the H_2 constraint and determine the corresponding value of λ, it is much easier in the optimization process to vary λ for fixed γ. Thus, the overbound for the infinity norm is held fixed, and the H_2 norm approaches a lower limit as λ is increased. This occurs when the H_∞ norm of the closed-loop system approaches the overbound γ. Thus, for any fixed value of γ, the procedure is mathematically equivalent to minimizing the H_∞ norm subject to a constraint on the H_2 norm. That is, γ is the smallest achievable H_∞ norm when the H_2 norm is constrained to be the lowest value achieved by fixing γ and increasing λ. Hence, the potential gap that exists in other formulations is removed. A method for finding solutions to these equations is outlined next.

Solution Method

Minimization of the performance index J'_γ in Eq. (38) can be carried out using a conjugate gradient method with Eqs. (41–45). This algorithm requires as a starting point an initial stabilizing gain. An algorithm for searching for such a gain has been suggested.[24,25] The conjugate gradient algorithm is as follows.

0) Set $\lambda = 0$. Find G_0, which stabilizes the closed-loop system, and select γ sufficiently large so that $G_0 \in G_\gamma$. Set $i = 0$.

1) Solve Eq. (42) and then Eqs. (41), (43), and (44) with $G = G_i$. Compute the gradient $(\partial L/\partial G)$ as defined by Eq. (45); if $\partial L/\partial G < \varepsilon$, stop; otherwise, compute the search direction ΔG. On the first iteration, this is equal to the negative of the gradient; otherwise, the direction is given by

$$\Delta G_i = -\left(\frac{\partial L}{\partial G}\right)_i + \beta \Delta G_{i-1}$$

$$\beta = \left\| \left(\frac{\partial L}{\partial G}\right)_i \right\|_F \Bigg/ \left\| \left(\frac{\partial L}{\partial G}\right)_{i-1} \right\|_F$$

where $\|\cdot\|_F$ represents the Froebenius norm.

2) Perform a one-dimensional line search to minimize $J'_\gamma(G)$, where $G = G_i + \alpha \Delta G$ and α is varied. At each step, it is necessary to check that G stabilizes the closed-loop system and that the Riccati equation (42) has a positive semidefinite solution or, equivalently, that the Hamiltonian matrix has no imaginary eigenvalues. Note that the stable eigenvalues are those of the system with the loop also closed on the disturbance.

3) Let $G_{i+1} = G^*$, where G^* minimizes $J_\gamma(G)$ in the line search. Set $i = i + 1$, and go to step 1.

As stated, there are two parameters, λ and γ, which can be varied. If γ is fixed, the algorithm may be employed for successively larger values of λ, restarting the algorithm each time λ is increased with the result of the preceding optimization. This process is repeated until no further reduction in the H_2 norm is observed. The resulting controller may be interpreted as providing the smallest H_2 norm such that the H_∞ norm is less than γ.

This approach differs from that of Walker and Ridgely[10] in that they use the H_2 controller as the "central controller," with an equality constraint on the infinity norm, whereas here the central controller is a suboptimal H_∞ controller, and the constraint is on the H_2 norm. This difference may have important computational significance because constraining the H_∞ norm implies a search for the largest value of γ such that the Hamiltonian matrix associated with Eq. (42) has eigenvalues on the imaginary axis. In the formulation, γ always serves as an overbound for the H_∞ norm. Also, the H_∞ performance in Ref. 10 can be formed with no penalty on control. In contrast, the H_2 performance can be formed with no penalty on control with the approach in this paper.

Examples

Example 1

A block diagram of the generalized plant is shown in Fig. 4. For the H_∞ problem, the input is $w = w_1$ and the outputs are $z = [z_1 \ z_2 \ z_3]^T$. For the H_2 problem, the inputs are $w_p = [w_{d1} \ w_{d2}]^T$ and the outputs are $z_p = [z_{p1} \ z_{p2}]^T$. The measurement and control signals are y and u, respectively. The various blocks are defined as follows:

$$P = \frac{1}{s+1}, \qquad W_1 = \frac{10}{s+10}$$

$$W_2 = 0.01, \qquad W_3 = \frac{10(s+10)}{s+100}, \qquad W_{d1} = 10$$

$$W_{d2} = 1, \qquad W_{p1} = 10, \qquad W_{p2} = 1$$

Dynamic compensators of orders 3 and 1 were designed for various values of g using the algorithm described in the preceding section. Full-order H_∞ controllers also were designed for different values of γ using the two-Riccati-equation method.[15] One set of H_∞ controllers was designed with only the input w and the output z, whereas the other set was designed with all inputs w and w_p and all outputs z and z_p. The infinity norms of all of the designs were evaluated for T_{zw}, and the two norms of the controllers and compensators were evaluated for $T_{z_p w_p}$. The results are plotted in Fig. 5

In the full-order case, the mixed design shows a significantly lower H_2 norm than the H_∞ controller without the H_2 variables. With the H_2-norm variables included in the H_∞ design, the solutions for all values of g are equivalent to the full-order H_2 optimal solution. The first-order design exhibits significant degradation in H_2 performance for infinity norms below 0.95. At each design point, i.e., each value of γ, it was observed that the true H_∞ norm of the fixed-order designs approached the overbound γ as λ was increased, indicating that no further reduction of the H_2 norm was possible without violating the infinity norm constraint. As expected,

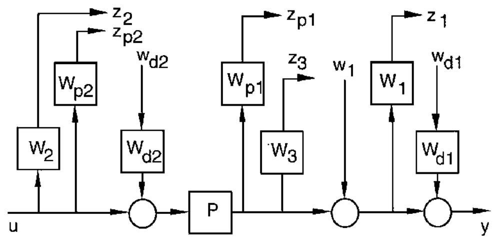

Fig. 4 Block diagram for third-order mixed problem.

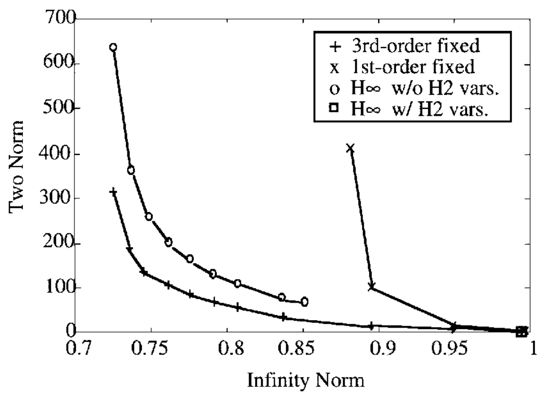

Fig. 5 Performance tradeoff for example 1.

the H_2 performance of the H_∞ controller degrades significantly as the optimum H_∞ norm is approached.

Example 2

Example 2 is a model of the lateral dynamics of the AH-64 Apache helicopter in forward flight with $V = 200$ ft/s and $h = 100$ m. The objective is to design a bank-to-turn flight controller, where the pilot inputs a bank command and the controller maintains not only the desired bank angle but also the appropriate yaw rate such that sideslip is near zero, i.e., the controller produces a coordinated turn. The generalized plant is shown schematically in Fig. 6. The H_2 problem is defined by the input w_p and the output z_p, and the H_∞ problem is defined by the inputs w_1, w_2, d_1, d_2, and d_3 and the outputs z_1, z_2, z_3, and z_4.

The H_2 problem is to minimize sideslip due to lateral gusts, and the H_∞ problem is to achieve good tracking and also provide robustness to uncertainty at the plant input. Using white noise w_p and the Dryden turbulence model,[26] $W_g(s)$, a lateral gust disturbance was included in the model and serves as the input for the H_2 problem. A lateral wind gust v_g also was used as an $\boldsymbol{L}_2$ disturbance for the H_∞ problem, and sensor noise was assumed to be present, denoted by d_1, d_2, and d_3. Multiplicative input uncertainty was included to represent unmodeled actuator and rotor dynamics, scaled by W_Δ. A frequency-dependent weight W_e was placed on the bank error to create integral-type control, and constant weights were placed on u, v, p, and r to limit magnitudes. Lateral acceleration (a_y) was used as the H_2 performance variable, indicated by z_p, where

$$\begin{aligned} a_y &= \dot{v} + u_0 r - w_0 p + g\phi \\ &= Y_v(v - v_g) + Y_p p + Y_r r + Y_{\delta_a}\delta_a + Y_{\delta_r}\delta_r \end{aligned}$$

Y_i are the stability derivatives, and δ_a and δ_r are the lateral stick and pedal displacements, respectively. The order of the generalized plant is 10. The lateral dynamics are given by (A, B, C), where B includes the control input matrix and the gust input matrix:

$$A = \begin{bmatrix} -0.19043 & -8.85648 & -200.122 & 32.1304 \\ -0.02906 & -3.51653 & 0.17469 & 0.00000 \\ -0.01642 & -0.19374 & -0.95976 & 0.00000 \\ 0.00000 & 1.00000 & -0.02419 & 0.00000 \end{bmatrix}$$

$$B = \begin{bmatrix} 1.3054 & 2.3916 & 0.19043 \\ 1.4594 & 0.7029 & 0.02906 \\ 0.01729 & -0.8338 & -0.01642 \\ 0.0000 & 0.0000 & 0.00000 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$z_p = W_{p1}x + W_{p2}u + W_{p3}V_g$$

where

$$W_{p1} = [-0.1904 \quad 0.9557 \quad 2.2678 \quad 0]$$

$$W_{p2} = [1.3054 \quad 2.3916], \qquad W_{p3} = [0.19034]$$

$$W_{\phi c} = \frac{10}{s+10}, \qquad W_g = \frac{8.6419(s + 0.4755)}{(s^2 + 1.6472s + 0.6783)}$$

$$W_d = 0.01, \qquad W_v = \frac{1}{50}, \qquad W_p = \frac{0.1}{50}$$

$$W_r = \frac{0.1}{50}, \qquad W_e = \frac{1}{50}\frac{0.4(s+100)}{s+0.01}$$

$$W_\Delta = \frac{s+10}{s+100} \times I_2, \qquad W_u = \frac{1}{50}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Figure 7 shows the tradeoff in performance when designs are carried out for varying values of γ. Seventh-order and third-order compensators were designed as well as full-order H_∞ controllers with and without the H_2 problem variables w_p and z_p. The norms of the closed-loop transfer functions T_{zw} and $T_{z_p w_p}$ were evaluated as in example 1. It can be seen that the seventh-order mixed compensator has an improved H_2 norm over the H_∞ controller designed without H_2 performance. When the H_2 variables w_p and z_p are not included in the H_∞ design, the H_2 norm can be improved by relaxing the infinity-norm constraint. When the H_2 variables are included, the H_2 performance and the infinity norm both increase. It also can be seen that the mixed designs allow a wider range of tradeoff between the two criteria. The third-order compensator performance actually dips below that of the seventh-order compensator at higher values of the infinity norm. This, though, may be an indication that the algorithm has converged on a local minimum.

Figures 8 and 9 show the time responses of the closed-loop system to a step input in bank angle. Both the seventh-order and the third-order compensators have responses close to that of the full-order H_∞ controller designed without including w_p and z_p. When w_p and z_p are included, the performance decreases significantly. The lateral velocity with the seventh-order compensator is slightly higher than with the third-order design.

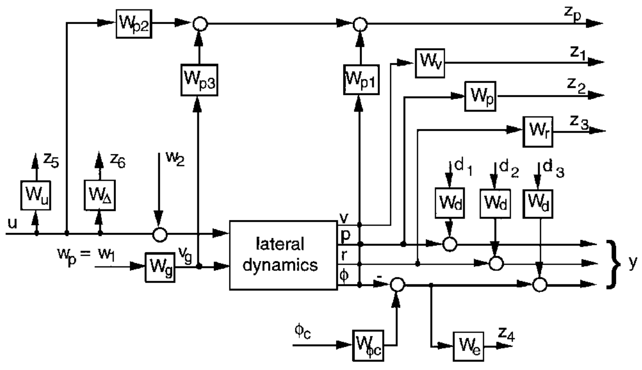

Fig. 6 Generalized plant for Apache forward-flight mixed problem.

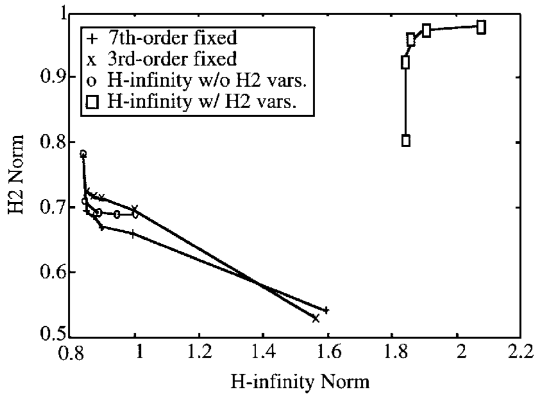

Fig. 7 Performance tradeoff for example 2.

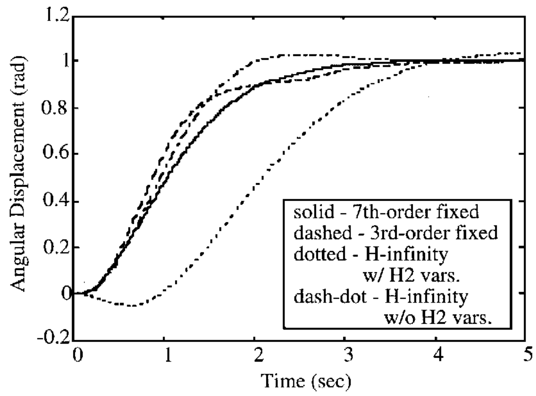

Fig. 8 Bank-angle response to step command and lateral gust.

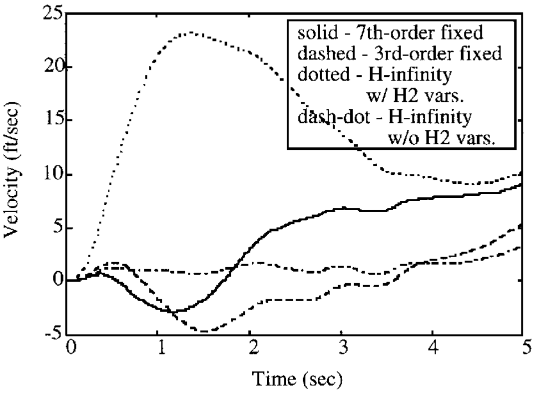

Fig. 9 Lateral velocity due to bank command and lateral gust.

Conclusions

A method of synthesizing fixed-order mixed H_∞ controllers is presented. The approach employs a canonical compensator structure and is based on a differential game formulation. The mixed problem requires only minimal modifications to the H_∞ problem formulation. The example indicates significant gains in H_2 performance over designs based on minimizing the H_∞ norm.

Acknowledgments

This research was supported by the Army Research Office under Grant DAAH04-93-G-0002 and by the National Science Foundation under Grant BCS-9202039.

References

[1]Glover, K., "All Optimal Hankel-Norm Approximations of Linear Multivariable Systems and Their L-Infinity Error Bounds," *International Journal of Control*, Vol. 39, No. 6, 1984, pp. 1115–1193.

[2]Bernstein, D. S., and Haddad, W. M., "LQG Control with an H_∞ Performance Bound: A Riccati Equation Approach," *IEEE Transactions on Automatic Control*, Vol. AC-34, No. 3, 1989, pp. 293–305.

[3]Doyle, J., Zhou, K., and Bodenheimer, B., "Optimal Control with Mixed H_2 and H_∞ Performance Objectives," *Proceedings of the 1989 American Control Conference* (Pittsburgh, PA), American Automatic Control Council, 1989, pp. 2065–2070.

[4]Hyland, D. C., and Bernstein, D. S., "The Optimal Projection Equations for Fixed-Order Dynamic Compensators," *IEEE Transactions on Automatic Control*, Vol. AC-29, No. 11, 1984, pp. 1034–1037.

[5]Zhou, K., Doyle, J., Glover, K., and Bodenheimer, B., "Mixed H_2 and H_∞ Control," *Proceedings of the 1990 American Control Conference* (San Diego, CA), 1990, pp. 2502–2507.

[6]Khargonekar, P. P., and Rotea, M. A., "Mixed H_2/H_∞ Control: A Convex Optimization Problem," *IEEE Transactions on Automatic Control*, Vol. AC-36, No. 7, 1991, pp. 824–837.

[7]Rotea, M., and Khargonekar, P., "H_2 Optimal Control with an H_∞ Constraint: The State Feedback Case," *Automatica*, Vol. 27, No. 2, 1991, pp. 307–316.

[8]Ridgely, D. B., "A Nonconservative Solution to the General Mixed H_2/H_∞ Optimization Problem," Ph.D. Thesis, Dept. of Aeronautics and Astronautics, Massachusetts Inst. of Technology, Cambridge, MA, 1992.

[9]Ridgely, D. B., Valvani, L., Dahleh, M., and Stein, G., "Solution to the General Mixed H_2/H-Infinity Control Problem—Necessary Conditions for Optimality," *Proceedings of the American Control Conference* (Chicago, IL), 1992, pp. 1348–1352.

[10]Walker, D. E., and Ridgely, D. B., "Reduced Order Mixed H_2/H_∞ Optimization with a Singular H_∞ Constraint," *Proceedings of the American Control Conference* (Baltimore, MD), 1994, pp. 1128–1132.

[11]Mageirou, E. F., "Values and Strategies for Infinite Time Linear Quadratic Games," *IEEE Transactions on Automatic Control*, Vol. AC-21, No. 8, 1976, pp. 547–550.

[12]Magierou, E. F., and Ho, Y. C., "Decentralized Stabilization via Game Theoretic Methods," *Automatica*, Vol. 13, No. 4, 1977, pp. 393–399.

[13]Papavassilopoulos, G. P., and Safonov, M. G., "Robust Control Design via Game Theoretic Methods," *Proceedings of 28th IEEE Conference on Decision and Control* (Tampa, FL), Inst. of Electrical and Electronics Engineers, New York, 1989, pp. 382–387.

[14]Rhee, I., and Speyer, J. L., "A Game-Theoretic Approach to a Finite-Time Disturbance Attenuation Problem," *IEEE Transactions on Automatic Control*, Vol. AC-36, No. 9, 1991, pp. 1021–1032.

[15]Doyle, J. C., Glover, K., Khargonekar, P. P., and Francis, B. A., "State-Space Solutions to Standard H_2 and H_∞ Control Problems," *IEEE Transactions on Automatic Control*, Vol. AC-34, No. 8, 1989, pp. 831–847.

[16]Basar, T., and Bernhard, P., *H-Infinity Optimal Control and Related Minimax Design Problems, A Dynamic Game Approach*, Birhäuser, Boston, 1991.

[17]El Ghaoui, L., Carrier, A., and Bryson, A. E., "Linear Quadratic Minimax Controllers," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 4, 1992, pp. 953–961.

[18]Sweriduk, G. D., and Calise, A. J., "Fixed-Order Robust Control: A Differential Game Approach," *1st IEEE Conference on Aerospace Systems* (Westlake Village, CA), Inst. of Electrical and Electronics Engineers, New York, 1993, pp. 458–462.

[19]Theodor, Y., and Shaked, U., "Output Feedback Mixed H_2/H_∞ Control—A Dynamic Game Approach," *International Journal of Control*, Vol. 64, No. 2, 1996, pp. 263–279.

[20]Kramer, F. S., and Calise, A. J., "Fixed Order Dynamic Compensation for Multivariable Linear Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 11, No. 1, 1988, pp. 80–85.

[21]Byrns, E. V., Sweriduk, G. D., and Calise, A. J., "Optimal H_2 and H_∞ Fixed Order Dynamic Compensation Using Canonical Forms," *International Journal of Robust and Nonlinear Control*, Vol. 2, No. 4, 1992, pp. 243–260.

[22]Makila, P. M., and Toivonen, H. T., "Computational Methods for Parametric LQ Problems—A Survey," *IEEE Transactions on Automatic Control*, Vol. AC-32, No. 8, 1987, pp. 658–671.

[23]Makila, P. M., "Linear Quadratic Design of Structure-Constrained Controllers," *Proceedings of the 1983 American Control Conference* (San Francisco), 1983, pp. 1011–1019.

[24]Moerder, D. D., and Calise, A. J., "Convergence of a Numerical Algorithm for Calculating Optimal Output Feedback Gains," *IEEE Transactions on Automatic Control*, Vol. AC-30, No. 9, 1985, pp. 900–903.

[25]Collins, E. G., Jr., Haddad, W. M., and Ying, S. S., "Nearly Nonminimal Linear-Quadratic-Gaussian Compensators for Reduced-Order Control Design Initialization," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 1, 1996, pp. 259–261.

[26]Turner, R. E., and Hill, K., "Terrestrial Environment (Climatic) Criteria Guidelines for Use in Aerospace Vehicle Development," NASA TM 82473, 1982.